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AFDELING NUMERIEKE WISKUNDE NW 115/81 NOVEMBER
(DEPARTMENT OF NUMERICAL MATHEMATICS)

H.J.J. TE RIELE

COLLOCATION METHODS FOR WEAKLY SINGULAR SECOND KIND VOLTERRA INTEGRAL EQUATIONS WITH NON-SMOOTH SOLUTION

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Collocation methods for weakly singular second kind Volterra integral equations with non-smooth solution*)

by

Herman J.J. te Riele

ABSTRACT

Collocation type methods are studied for the numerical solution of the weakly singular Volterra integral equation of the second kind:

$$(1) \qquad f(t) = g(t) + \int_0^t K(s,f(s))(t-s)^{-\frac{1}{2}}ds, \quad t \in [0,T],$$

where the solution $f(t)$ is assumed to have the form $f(t) = \chi(t) + t^{\frac{1}{2}}\psi(t)$, χ and ψ being sufficiently smooth. The solution is approximated near zero by a linear combination of powers of $t^{\frac{1}{2}}$, and away from zero by the usual polynomial representation. Convergence is proved and many numerical experiments are carried out with examples from the literature. A comparison is made with a method of Brunner et al., originally developed for (1) with a *smooth* solution. Special attention is paid to the numerical approximation of the so-called moment integrals which emerge in the collocation scheme.

KEY WORDS & PHRASES: *weakly singular Volterra integral equation of the second kind, collocation method*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we consider the weakly singular Volterra integral equation of the second kind

$$(1.1) \qquad f(t) = g(t) + \int_0^t K(s,f(s))(t-s)^{-\frac{1}{2}}ds, \qquad t \in [0,T] =: I.$$

Throughout, we assume the given functions g and K to be sufficiently smooth, in order to guarantee the existence and unicity of a solution $f \in C(I)$ (see, for example, YOSIDA [19] and MILLER and FELDSTEIN [13]). Equations of the type (1.1) arise in the study of various problems in physics and chemistry, like heat conduction, superfluidity, electrochemistry and crystal growth (cf. CHAMBRÉ [4], KELLER and OLMSTEAD [8], LEVINSON [9], GHEZ and LEW [5], MANN and WOLF [12], NICHOLSON [14] and PADMAVALLY [18]). It should be remarked that usually the more general equation

$$(1.1') \qquad f(t) = g(t) + \int_0^t K(t,s,f(s))(t-s)^{-\alpha}ds, \qquad 0 < \alpha < 1,$$

is the starting point in numerical papers on weakly singular second kind Volterra integral equations. However, in all practical problems we have seen, K only depends on s and f, or even on f alone, and $\alpha = \frac{1}{2}$. This motivated our choice (1.1). Moreover, the extension of the present work to (1.1') is straightforward and does not present any fundamental difficulty.

The concept of product integration, developed by YOUNG [20,21], is often used in the numerical solution of (1.1): the function $K(s,f(s))$ is approximated by a linear combination of "simple" basis functions $a_i(s)$, say, with the property that the resulting so-called moment integrals $\int_0^t a_i(s)(t-s)^{-\frac{1}{2}}ds$ can be evaluated analytically for any $t \in I$. Another way of product integration is to approximate the solution f, instead of $K(s,f(s))$, by a linear combination of basis functions. This also yields moment integrals which, in general, cannot be evaluated analytically, and have to be approximated by some numerical quadrature formula. After this approximation of K resp. f, substitution of some discrete set of t-values t_i, $i = 1(1)N$, gives a triangular system of equations in $f_1,f_2,\ldots,f_N$ (where f_i denotes a numerical approximation of $f(t_i)$), resp. in the coefficients of the linear combination by which $f(t)$ was approximated. See NOBLE [15], LINZ [10], DE HOOG and WEISS [7] and LOGAN [11].

BRUNNER and EVANS [2] and BRUNNER and NØRSETT [3] have studied so-called collocation methods for (1.1), which are related to, and, in certain cases, reduce to product integration methods. In these collocation methods the exact solution of (1.1) is projected into the space of piecewise polynomials of degree $m \geq 1$ with prescribed knots. Here, it is assumed that the solution of (1.1) is sufficiently smooth on I. However, according to an extensive study by MILLER and FELDSTEIN [13] of the smoothness of solutions of (1.1), $f(t)$ is, in general, *not* smooth in the neighbourhood of $t = 0$. In particular, one usually has $f'(t) = \mathcal{O}(t^{-\frac{1}{2}})$, as $t \to 0$. This may be illustrated by the typical example:

$$(1.2) \qquad f(t) = 1 - \int_0^t (t-s)^{-\frac{1}{2}} f(s)\,ds,$$

with solution $f(t) = \exp(\pi t)\operatorname{erfc}(\pi^{\frac{1}{2}} t^{\frac{1}{2}}) = 1 - 2t^{\frac{1}{2}} + \mathcal{O}(t)$, as $t \to 0$, where $\operatorname{erfc}(t) = 1 - \operatorname{erf}(t)$, erf being the error function ([1, p.297]).

In this paper we shall study certain collocation methods for solving (1.1), where the solution is assumed to be of the form

$$(1.3) \qquad f(t) = \chi(t) + t^{\frac{1}{2}}\psi(t),$$

χ and ψ being sufficiently smooth functions on I. In order to deal with this behaviour of f, it is approximated near zero not by polynomials, but by powers of $t^{\frac{1}{2}}$. This gives a considerable increase of accuracy compared with polynomial collocation methods for (1.1) with solution of the form (1.3). We shall also pay special attention to the numerical evaluation of the moment integrals which arise in these methods, for the case one does not want to, or cannot evaluate them analytically.

2. COLLOCATION METHODS FOR SOLVING (1.1)

For a given $N \in \mathbb{N}$ let $h := T/N$, $t_i := ih$ $(i = 0(1)N)$ and $\sigma_i := (t_i, t_{i+1}]$. The unique solution $f(t)$ of (1.1) will be approximated by an element $u \in S_m$, where $m \in \mathbb{N}$ is given and

$$(2.1)\qquad S_m := \{u(t),\ u\big|_{t\in\sigma_k} =: u_k(t) = \sum_{i=0}^{m} a_{ki}\phi_{ki}(t),\ k = 0(1)N-1\}.$$

It turns out that the basis functions ϕ_{ki} play a crucial role. Their choice should be such that f(t) can be approximated properly by an element of S_m, in particular for t near zero. We consider two choices denoted by A and B:

$$(2.2)\qquad \begin{cases} A\ \ :\ \phi_{0i}(t) := [(t-t_0)/h]^{i/2} \\ B\ \ :\ \phi_{0i}(t) := [(t-t_0)/h]^{i} \end{cases},\quad i = 0(1)m;$$
$$\qquad\qquad A\&B:\ \phi_{ki}(t) := [(t-t_k)/h]^{i},\quad i = 0(1)m,\ \ k = 1(1)N-1.$$

For these choices the space S_m consists of piecewise continuous functions with, at most, $N-1$ finite discontinuities in $t_1,t_2,\ldots,t_{N-1}$. We define a collocation set X by

$$(2.3)\qquad X := \bigcup_{k=0}^{N-1} X_k,\quad \text{where } X_k = \{t = t_{kj} := t_k + \eta_j h,\ j = 0(1)m\},$$

with $0 < \eta_0 < \eta_1 < \ldots < \eta_{m-1} < \eta_m = 1$. The fixed numbers η_j are called the collocation parameters. Now an approximation $u \in S_m$ of the exact solution f(t) of (1.1) is sought by projecting f into the space S_m, i.e., by requiring that

$$(2.4)\qquad u(t) = g(t) + \int_0^t K(s,u(s))(t-s)^{-\frac{1}{2}}ds,\quad \text{for } t \in X.$$

The existence of a unique solution of (2.4) for all sufficiently small $h > 0$ can be proved with a contraction mapping argument, under sufficient smoothness of K and g (cf. BRUNNER and EVANS [2]). For simplicity of presentation we now proceed with the assumption that K is linear in f, i.e., $K(s,f) = K(s)f$. The function $u \in S_m$ will be computed in a recursive way: assuming the pieces $u_0(t),\ldots,u_{k-1}(t)$ of u(t) to be known, the next piece $u_k(t)$ is computed from the equation

$$(2.4')\qquad u_k(t) - \int_{t_k}^{t} K(s)(t-s)^{-\frac{1}{2}}u_k(s)ds = g(t) + \int_{t_0}^{t_k} K(s)(t-s)^{-\frac{1}{2}}u(s)ds,$$
$$t \in X_k;\ k = 0(1)N-1.$$

With the representation of u in (2.1) and the definition of X_k in (2.3) this yields the linear system of equations in a_{ki}:

(2.5) $$\sum_{i=0}^{m} a_{ki}[\phi_{ki}(t_{kj}) - M_{kij}] = g(t_{kj}) + N_{kj}, \quad j = 0(1)m;\ k = 0(1)N-1,$$

where M_{kij} and N_{kj} are the so-called moment integrals, defined by

(2.6) $$M_{kij} := \int_{t_k}^{t_{kj}} K(s)\phi_{ki}(s)(t_{kj} - s)^{-\frac{1}{2}} ds,$$

and

(2.7) $$N_{kj} := \int_{t_0}^{t_k} K(s)(t_{kj} - s)^{-\frac{1}{2}} u(s) ds.$$

Since

$$\phi_{ki}(t_{kj}) = \begin{cases} \eta_j^{i/2} & \text{for choice A, } k = 0 \text{ (see (2.2)),} \\ \eta_j^{i} & \text{otherwise,} \end{cases}$$

and since M_{kij} can be made arbitrarily small for sufficiently small $h > 0$, the matrix of coefficients of the linear system (2.5) is a "perturbed" Vandermonde matrix. Consequently, (2.5) has a unique solution for any sufficiently small $h > 0$ (cf. ORTEGA [16, p.32]).

In the sequel we shall denote the scheme (2.5) with the basis-functions $[(t-t_0)/h]^{i/2}$ on $(0,h]$ by scheme A and the other scheme by scheme B. Scheme B corresponds to a scheme described by BRUNNER and NØRSETT [3] in terms of (generalized) Radau abscissas. They also describe another scheme based on (generalized) Lobatto abscissas. The experiment which BRUNNER and NØRSETT show in [3], and some other experiments which we have carried out, reveal that their first scheme usually gives more accurate results than the second. Therefore, we decided to compare our scheme only with their first scheme (which we named scheme B).

The examples of (1.1) treated numerically in the literature are often constructed in such a way that the moment integrals in (2.5) can be evaluated analytically. In practical cases, however, this may not be possible, and therefore we shall compute them numerically. This will be described for M_{kij}

and N_{kj} separately, in the next two subsections.

2.1. Numerical computation of M_{kij}

Substitution of (2.2) into (2.6) yields with the transformation $\tau = (s-t_k)/(\eta_j h)$

$$(2.8)\qquad M_{kij} = h^{\frac{1}{2}} \begin{bmatrix} \eta_j^{(i+1)/2} \\ \\ \eta_j^{(2i+1)/2} \end{bmatrix} \int_0^1 K(t_k+\eta_j h\tau) \begin{bmatrix} \tau^{i/2} \\ \\ \tau^i \end{bmatrix} (1-\tau)^{-\frac{1}{2}} d\tau,$$

for the choice A, k = 0 resp. otherwise. The (weakly singular) integral will be approximated by using weighted interpolatory quadrature with abscissas $\eta_0,\eta_1,\ldots,\eta_m$ and weights $w_0,w_1,\ldots,w_m$ and with weight function $(1-\tau)^{-\frac{1}{2}}$. We notice that the quadrature abscissas are chosen such that they coincide with the collocation parameters in (2.3). Thus, the η_ℓ and w_ℓ are defined by

$$\int_0^1 v(\tau)(1-\tau)^{-\frac{1}{2}} d\tau = \sum_{\ell=0}^{m} w_\ell v(\eta_\ell) + E,$$

where E = 0 when $v(\tau) = \tau^{i/2}$ resp. τ^i for $i = 0(1)2m$. Since $\eta_m = 1$ is prescribed, we have $2m+1$ equations in the unknowns $\eta_0,\eta_1,\ldots,\eta_{m-1}$, $w_0,w_1,\ldots$ $\ldots,w_m$. In Table 2.1 we present the solutions for m = 1,2,3, for the two cases considered.

2.2. Numerical computation of N_{kj}

Substitution of (2.2) into (2.7) yields with (2.1) and the transformation $\tau = (s-t_k)/h$

$$(2.9)\qquad \begin{aligned} N_{kj} &= h^{\frac{1}{2}} \sum_{i=0}^{m} a_{0i} \int_0^1 K(t_0+h\tau) \begin{bmatrix} \tau^{i/2} \\ \\ \tau^i \end{bmatrix} (k+\eta_j-\tau)^{-\frac{1}{2}} d\tau \; + \\ &+ h^{\frac{1}{2}} \sum_{\ell=1}^{k-1} \sum_{i=0}^{m} a_{\ell i} \int_0^1 K(t_\ell+h\tau)\tau^i(k-\ell+\eta_j-\tau)^{-\frac{1}{2}} d\tau. \end{aligned}$$

TABLE 2.1.

Abscissas η_ℓ and weights w_ℓ of weighted interpolatory quadrature

$$\int_0^1 v(\tau)(1-\tau)^{-\frac{1}{2}}d\tau = \sum_{\ell=0}^{m} w_\ell v(\eta_\ell),$$

with $\eta_m = 1$ (prescribed).

Quadrature exact for $v(\tau) = \tau^{i/2}$, $i = 0(1)2m$

	m = 1	m = 2	m = 3
η_0	0.306101188813	0.089361483186	0.033732053372
w_0	0.960754876530	0.282943402907	0.108505616907
η_1	1.000000000000	0.595690441907	0.282593677396
w_1	1.039245123470	1.011619680159	0.469519838477
η_2		1.000000000000	0.746460414456
w_2		0.705436916870	0.887374907658
η_3			1.000000000000
w_3			0.534599636959

Quadrature exact for $v(\tau) = \tau^{i}$, $i = 0(1)2m$

	m = 1	m = 2	m = 3
η_0	0.400000000000	0.178838086815	0.099194170728
w_0	1.111111111111	0.473853770112	0.258969932338
η_1	1.000000000000	0.710050802074	0.450131500784
w_1	0.888888888889	0.957257340999	0.559410782978
η_2		1.000000000000	0.835289713103
w_2		0.568888888889	0.763660101011
η_3			1.000000000000
w_3			0.417959183673

Here, the integrand shows a weak singularity *off* the integration interval, which, however, is very near to it in the case $\ell = k-1$. As is well-known, this will affect the accuracy of a standard quadrature formula adversely. Therefore, the integral in (2.9) will be approximated, like the integral in (2.8), using weighted interpolatory quadrature, with weight function $(k-\ell+\eta_j-\tau)^{-\frac{1}{2}}$. At first sight, this appears to cause a considerable amount of work, since the weight function depends on k, ℓ and j so that many weights and abscissas are required. However, three arguments make this work acceptable:

(i) as will be clarified in Section 3, only 1- and 2-point weighted quadrature will be needed for N_{kj};

(ii) thanks to the convolution form of the factor $(t-s)^{-\frac{1}{2}}$ in (1.1) (which is reflected in the term $k-\ell$ in the weight function in (2.9)), our numerical scheme only requires the abscissas and weights of quadrature formulas with weight function $(y-\tau)^{-\frac{1}{2}}$ for $y = n+\eta_j$ for $j = 0(1)m$ and $n = 1(1)N-1$. Hence, before starting the numerical scheme, the N_{kj} require the computation and storage of $(m+1)(N-1)$ sets of weights and abscissas;

(iii) several numerical experiments with other quadrature formulas for (2.9) indicate that the weighted quadrature proposed here gives good results with respect to accuracy and computational effort.

For r-point weighted quadrature $(r = 1,2)$ and for a given value of $y > 1$ the corresponding weights and abscissas $w_i(y)$ and $\eta_i(y)$ $(i = 1(1)r)$ are defined by the equations

$$(2.10)\qquad \int_0^1 v(\tau)(y-\tau)^{-\frac{1}{2}}d\tau = \sum_{i=1}^{r} w_i(y)v(\eta_i(y)) + E,$$

where $E = 0$ when $v(\tau) = \tau^{i/2}$ resp. τ^i for $i = 0(1)2r-1$. We notice that here none of the abscissas is prescribed a priori, as contrasted with the computation of M_{kij} in subsection 2.1. Since only $r = 1$ or $r = 2$, *explicit* formulas can be derived for the computation of $w_i(y)$ and $\eta_i(y)$ from (2.10) (cf. [6, p. 422]). Values of the functions $J_i(y) := \int_0^1 \tau^{i/2}(y-\tau)^{-\frac{1}{2}}d\tau$ are needed, for $i = 0,1,2,3,4$ and 6. Explicit forms are given in Table 2.2 below.

TABLE 2.2

i	$J_i(y) = \int_0^1 \tau^{i/2}(y-\tau)^{-\frac{1}{2}}d\tau$
0	$2[y^{\frac{1}{2}} - (y-1)^{\frac{1}{2}}]$
1	$y \arcsin(y^{-\frac{1}{2}}) - (y-1)^{\frac{1}{2}}$
2	$\frac{4}{3}[y^{3/2} - (y-1)^{3/2}] - 2(y-1)^{\frac{1}{2}}$
3	$\frac{3}{4}y^2 \arcsin(y^{-\frac{1}{2}}) - \frac{1}{4}(3y+2)(y-1)^{\frac{1}{2}}$
4	$\frac{16}{15}[y^{5/2} - (y-1)^{5/2}] - \frac{8}{3}(y-1)^{3/2} - 2(y-1)^{\frac{1}{2}}$
6	$\frac{32}{35}[y^{7/2} - (y-1)^{7/2}] - \frac{16}{5}(y-1)^{5/2} - 4(y-1)^{3/2} - 2(y-1)^{\frac{1}{2}}$

3. CONVERGENCE

We shall present here a convergence theorem for the schemes A and B where $f(t)$ is of the form (1.3). The error is denoted by

$$e(t) := f(t) - u(t), \quad t \in I, \tag{3.1}$$

where $u(t)$ is the approximation of f, found by either scheme A or scheme B. By subtraction of (2.4) from (1.1) it follows that $e(t)$ satisfies

$$e(t) = \int_0^t K(s)(t-s)^{-\frac{1}{2}}e(s)ds, \quad t \in X. \tag{3.2}$$

THEOREM 3.1. *If* $f(t)$ *satisfies* (1.3) *then for both scheme* A *and scheme* B *we have* $e(t) = O(h^{\frac{1}{2}})$, *as* $h \to 0_+$, $Nh = T$, *for all* $t \in I$, *provided that the moment integrals* (2.6) *and* (2.7) *are evaluated with sufficient accuracy.*

REMARK. The numerical experiments reported in Section 4 indicate that the actual order of the error may be higher than $\frac{1}{2}$. It is an open problem to determine the precise order of the error for both schemes. For *smooth* f we have the following

THEOREM 3.2. *If* $f(t)$ *is sufficiently smooth on* I *then we have for scheme* A *that* $e(t) = O(h^{(m+1)/2})$ *and for* B *that* $e(t) = O(h^{m+1})$, *as* $h \to 0_+$, $Nh = T$, *for all* $t \in I$, *provided that the moment integrals are evaluated with sufficient accuracy.*

This theorem was proved by BRUNNER and NØRSETT [3] for the scheme B. Their proof can be adapted easily for the scheme A.

REMARK. The error in the numerical computation of the moment integrals described in Subsections 2.1 and 2.2 is $\mathcal{O}(h^{2m+1})$ for M_{kij} and $\mathcal{O}(h^{2r})$ for N_{kj}, as $h \to 0_+$. Following BRUNNER and NØRSETT [3] we find that the condition $r \geq (m+1)/2$ gives sufficient accuracy of the moment integrals in Theorem 3.2, for the scheme B. Since the order m+1 is the highest obtainable we shall choose r in all our experiments such that this condition is satisfied.

PROOF OF THEOREM 3.1. Consider first the scheme A, $t \in (0,h]$. We expand χ and ψ in (1.3) into a Taylor series near the origin to give

$$(3.3) \qquad f(t) = \sum_{i=0}^{m} b_{0i}\phi_{0i}(t) + R_0(t),$$

where

$$\phi_{0i}(t) = (t/h)^{i/2}, \qquad i = 0(1)m,$$

$$\left.\begin{array}{l} b_{0,2\ell} = h^{\ell}\chi^{(\ell)}(0)/\ell! \\ b_{0,2\ell+1} = h^{\ell+\frac{1}{2}}\psi^{(\ell)}(0)/\ell! \end{array}\right\} \ell = 0,1,\ldots ,$$

and

$$(3.4) \qquad R_0(t) = \mathcal{O}(h^{(m+1)/2}), \qquad \text{as } h \to 0_+.$$

On σ_0 we have $u(t) = u_0(t) = \Sigma_{i=0}^{m} a_{0i}\phi_{0i}(t)$, so that the error can be written as

$$(3.5) \qquad e(t) := e_0(t) = h^{(m+1)/2} \sum_{i=0}^{m} c_{0i}\phi_{0i}(t) + R_0(t), \qquad t \in \sigma_0,$$

where we have set $h^{(m+1)/2}c_{0i} := b_{0i} - a_{0i}$, $i = 0(1)m$. By substituting (3.5) into (3.2) (for $t = \eta_j h$, $j = 0(1)m$), dividing by $h^{(m+1)/2}$ and using the transformation $s = h\tau$ in the integrals, we obtain the system

$$(3.6)\qquad \begin{aligned}&\sum_{i=0}^{m} c_{0i}\left(\eta_j^{i/2} - h^{\frac{1}{2}}\int_0^{\eta_j} K(h\tau)(\eta_j-\tau)^{-\frac{1}{2}}\tau^{i/2}d\tau\right)\\ &= h^{-(m+1)/2}\left(-R_0(\eta_j h) + h^{\frac{1}{2}}\int_0^{\eta_j} K(h\tau)(\eta_j-\tau)^{-\frac{1}{2}}R_0(h\tau)d\tau\right),\qquad j = 0(1)m.\end{aligned}$$

Similarly as in Section 2, equation (2.5), this system has a unique solution for all sufficiently small $h > 0$. By using (3.4) one easily shows that the right hand side of (3.6) is bounded as $h \to 0_+$, so that also the c_{0i} are bounded, uniformly in i. From (3.5) and, again, (3.4) it follows that

$$(3.7)\qquad e_0(t) = \mathcal{O}(h^{(m+1)/2}),\quad \text{as } h \to 0_+,$$

on σ_0, for the scheme A. For the scheme B, which uses on σ_0 the basis functions $(t/h)^i$ instead of $(t/h)^{i/2}$, we can only prove (when $f(t)$ is of the form (1.3)) in an analogous way (with the difference that $R_0(t) = \mathcal{O}(h^{\frac{1}{2}})$ as $h \to 0_+$) that

$$(3.8)\qquad e_0(t) = \mathcal{O}(h^{\frac{1}{2}}),\quad \text{as } h \to 0_+,$$

on σ_0.

Next, we consider both schemes A and B, on σ_k, $k \geq 1$. Expanding f into a Taylor series near $t_k = kh$ yields

$$(3.9)\qquad f(t) = \sum_{i=0}^{m} b_{ki}\phi_{ki}(t) + R_k(t),$$

where

$$\left.\begin{aligned}\phi_{ki}(t) &= [(t-t_k)/h]^i\\ b_{ki} &= h^i f^{(i)}(t_k)/i!\end{aligned}\right\}\quad i = 0(1)m,$$

and

$$(3.10)\qquad R_k(t) = h^{m+1}\phi_{k,m+1}(t)f^{(m+1)}(t_k + \theta_k(t))/(m+1)!$$

for some $\theta_k(t)$ with $t_k < \theta_k(t) < t$. On σ_k we have $u(t) = u_k(t) = \Sigma_{i=0}^m a_{ki}\phi_{ki}(t)$, so that for the error we can write

$$e(t) := e_k(t) = h^{\frac{1}{2}} \sum_{i=0}^{m} c_{ki}\phi_{ki}(t) + R_k(t), \quad t \in \sigma_k, \tag{3.11}$$

where we have set $h^{\frac{1}{2}}c_{ki} := b_{ki} - a_{ki}$, $i = 0(1)m$. Now for $j = 0(1)m$ we set $t := t_{kj} = (k+\eta_j)h$ in (3.2) and subtract from the resulting equation the one which we obtain by setting $t := t_{k-1,m}$ $(=t_k)$ in (3.2). This gives

$$e_k(t_{kj}) = e_{k-1}(t_{k-1,m}) + \int_0^{t_{kj}} K(s)(t_{kj}-s)^{-\frac{1}{2}}e(s)ds$$
$$- \int_0^{t_k} K(s)(t_{k-1,m}-s)^{-\frac{1}{2}}e(s)ds, \quad j = 0(1)m.$$

By splitting the integrals into pieces and using (3.11), we obtain for the scheme B in terms of the $c_{\ell i}$, $\ell = 0(1)k$, $i = 0(1)m$:

$$\sum_{i=0}^{m} c_{ki}\{\eta_j^i - h^{\frac{1}{2}} \int_0^{\eta_j} K(t_k+h\tau)(\eta_j-\tau)^{-\frac{1}{2}}\tau^i d\tau\}$$

$$= \sum_{i=0}^{m} c_{k-1,i}[1 + h^{\frac{1}{2}} \int_0^1 K(t_{k-1}+h\tau)\tau^i\{(1+\eta_j-\tau)^{-\frac{1}{2}} - (1-\tau)^{-\frac{1}{2}}\}d\tau] +$$

$$+ h^{\frac{1}{2}} \sum_{\ell=0}^{k-2} \sum_{i=0}^{m} c_{\ell i} \int_0^1 K(t_\ell+h\tau)\tau^i\{(k+\eta_j-\ell-\tau)^{-\frac{1}{2}} - (k-\ell-\tau)^{-\frac{1}{2}}\}d\tau + \tag{3.12}$$

$$+ h^{-\frac{1}{2}}[-R_k(t_{kj}) + R_{k-1}(t_k) + h^{\frac{1}{2}} \int_0^{\eta_j} K(t_k+h\tau)(\eta_j-\tau)^{-\frac{1}{2}}R_k(t_k+h\tau)d\tau +$$

$$+ h^{\frac{1}{2}} \sum_{\ell=0}^{k-1} \int_0^1 K(t_\ell+h\tau)R_\ell(t_\ell+h\tau)\{(k+\eta_j-\ell-\tau)^{-\frac{1}{2}} - (k-\ell-\tau)^{-\frac{1}{2}}\}d\tau],$$

$$j = 0(1)m, \; k = 1(1)N-1.$$

For the scheme A one obtains a slightly more complicated system of linear equations in c_{ki} because one has to distinguish between σ_0, and σ_k for $k \geq 1$. We finish the proof now for B. The final part of the proof for A goes along the same lines.

Setting $c_k := (c_{k0}, c_{k1}, \ldots, c_{km})^T \in \mathbb{R}^{m+1}$ (3.12) may be written in matrix-vector form as

$$(3.12') \qquad c_k = A_k^{-1}(D_{k,k-1}c_{k-1} + h^{\frac{1}{2}} \sum_{\ell=0}^{k-2} D_{k\ell}\, c_\ell + u_k),$$

where A_k^{-1} exists for all sufficiently small $h > 0$. The matrices $D_{k\ell}$, $\ell = 0(1)k-1$ are defined in an obvious way; the vector u_k contains the terms of (3.12) which depend on R_ℓ, $\ell = 0(1)k$. In order to derive bounds on the norms of A_k, $D_{k\ell}$ and u_k we use the following easily provable facts:

(i) the mean value theorem for integrals: if $y(x)$ is a continuous, one-signed function on $[a,b]$ then $\int_a^b y(x)z(x)dx = z(\xi)\int_a^b y(x)dx$ for some $\xi \in [a,b]$;

(ii) $\int_0^1 \{(x-t)^{-\frac{1}{2}} - (x+\eta_j-t)^{-\frac{1}{2}}\}d\tau < 2x^{-3/2}$ for $x \geq 1$;

(iii) if f satisfies (1.3) then $|f^{(m+1)}(t)| < Ct^{-m-\frac{1}{2}}$ for any sufficiently small $t > 0$, hence from (3.10) it follows that

$$|R_\ell((\ell+\theta)h)| < C^* h^{m+1} f^{(m+1)}((\ell+\theta)h) < C^{**} h^{\frac{1}{2}} \ell^{-m-\frac{1}{2}},$$

for any sufficiently small $h > 0$, $0 < \theta < 1$, $\ell = 0(1)k$;

(iv) $\Sigma_{\ell=0}^{k-1} \ell^{-m-\frac{1}{2}}(k-\ell)^{-\frac{1}{2}} < Ck^{-3/2}$.

The constants involved do not depend on h.

Using (i) - (iv) it is not difficult to bound the norms of the matrices and vectors involved in (3.12'). Proceeding in the same way as BRUNNER and NØRSETT [3, p.355] it follows that the vectors c_k remain uniformly bounded as $h \to 0_+$, $Nh = T$. This implies by (3.11) and (3.10) that $e(t) = \mathcal{O}(h^{\frac{1}{2}})$, as $h \to 0_+$, $Nh = T$. □

4. NUMERICAL EXPERIMENTS

We have carried out extensive numerical experiments with the schemes A and B on various linear and nonlinear examples of (1.1). For the nonlinear problems the resulting nonlinear equations were solved with Newton's method. The experiments were programmed in ALGOL 68 and run on a CDC CYBER 750 computer system.

Table 4.1 gives a list of examples of (1.1) which we have found in the literature, supplied with a few simple linear and nonlinear test-problems. Examples 1, 2, 3a, 3c and 3e have a smooth solution $f(t)$, the other examples have a solution with an unbounded first or higher derivative in the origin.

Table 4.2 presents, for a selection of problems from Table 4.1, the quantity $-{}^{10}\log|f(T)-u(T)|$, obtained with the schemes A and B, for three combinations of m and r: $m = r = 1$; $m = r = 2$; $m = 3$, $r = 2$; for various values of h. It should be emphasized that for linear problems the schemes A and B require exactly the same amount of computational work for fixed values of m, r and h.

An inspection of the results shows that for linear problems with solution of the form (1.3) the scheme A is superior to B, especially for $m = 2$ and $m = 3$. This clearly illustrates, in the light of the remarks in Section 3 about the convergence properties of the schemes A and B in the first interval $(0,h]$, that the accuracy, obtainable in the endpoint T of the integration interval $(0,T]$, depends heavily on the accuracy of the computed solution in the first interval $(0,h]$. In this respect one could raise the question whether it would be profitable to use the basis functions $[(t-t_0)/h]^{i/2}$ not only on the first interval, but also on the second, third,... interval (in the form $[(t-t_1)/h]^{i/2}$, $[(t-t_2)/h]^{i/2}$,...). However, numerical experiments with the scheme A, modified in this way, show a loss of accuracy compared with our original scheme A.

For nonlinear problems, the picture is less transparent. For the simple problems 10a, 10b, 10c the scheme A gives better results than B, whereas, for problems 12 and 13, B performs slightly better than A.

As to the global convergence order of the error: for problems with solution of the form (1.3) the order obtained with the scheme A seems to increase with m, whereas the order obtained with B seems to be fixed on $\mathcal{O}(h^{3/2})$. This was confirmed by some additional experiments carried out with very small values of h.

For problems with a smooth solution the scheme B behaves roughly as one would expect on the ground of Theorem 3.1; the scheme A behaves in a similar way, although with less accuracy. Therefore, the advantage of A for non-smooth functions seem to be a disadvantage for smooth functions.

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TABLE 4.1. Examples of the equation $f(t) = g(t) + \int_0^t K(s,f(s))(t-s)^{-\frac{1}{2}}ds,\ t\in I.$

Linear problems: $K(s,f(s)) = K(s)f(s)$

#	f(t)	K(s)	g(t)	I	ref.
1.	$(1+t)^{-\frac{1}{2}}$	$-\frac{1}{4}$	$f(t)+\frac{\pi}{8}+\frac{1}{4}\arcsin(\frac{1-t}{1+t})$	[0,1]	[10]
2.	$\exp(-t)$	-1	$f(t)+2t^{\frac{1}{2}}M(1;\frac{3}{2};-t)$ *)	[0,3]	[3]
3a.	1	-1	$1+2t^{\frac{1}{2}}$	[0,1]	[13]
3b.	$t^{\frac{1}{2}}$	-1	$\frac{1}{2}\pi t+t^{\frac{1}{2}}$	[0,2]	
3c.	t	-1	$\frac{4}{3}t^{3/2}+t$	[0,2]	
3d.	$t^{3/2}$	-1	$\frac{3}{8}\pi t^2+t^{3/2}$	[0,2]	
3e.	t^2	-1	$\frac{16}{15}t^{5/2}+t^2$	[0,2]	
4.	$t^{\frac{1}{2}}$	$+1$	$-\frac{1}{2}\pi t+t^{\frac{1}{2}}$	[0,2]	[11]
5.	$1-\exp(\pi t)\mathrm{erfc}(\pi^{\frac{1}{2}}t^{\frac{1}{2}})$ **) $(=2t^{\frac{1}{2}}+O(t),\ t\to 0)$	-1	$2t^{\frac{1}{2}}$	[0,2]	[11,15]
6.	$\exp(\pi t)\mathrm{erfc}(\pi^{\frac{1}{2}}t^{\frac{1}{2}})$	-1	1	[0,1]	[13]
7.	$\Sigma_{k=1}^{\infty}\ (-1)^{k-1}t^{k/2}/\Gamma(k/2+1)$	$-\pi^{-\frac{1}{2}}$	$2\pi^{-\frac{1}{2}}t^{\frac{1}{2}}$	[0,2]	[17]
8.	$t^{\frac{1}{2}}\exp(-\alpha t)$	-1	$f(t)+\frac{1}{2}\pi t\exp(-\frac{1}{2}\alpha t)\cdot$ $\cdot\{I_0(-\frac{1}{2}\alpha t)+I_1(-\frac{1}{2}\alpha t)\}$ ***)	[0,3]	[11]
8a.	$\alpha=1$				
8b.	$\alpha=8$				
9.	$r(1+r^2)^{-1}\{\exp(r^2t)\mathrm{erfc}(rt^{\frac{1}{2}})$ $-\exp(-t)+2r\pi^{\frac{1}{2}}F(t^{\frac{1}{2}})\}$ $r=0.1,\ 1$ ****)	$-r\pi^{-\frac{1}{2}}$	$r(1-\exp(-t))$	[0,1]	[2,5]

*) M: Kummer's function [1, p.504]

**) erfc: complementary error function [1, p.297]

***) I_ν: modified Bessel function [1, p.375]

****) $F(t) = \exp(-t^2)\int_0^t \exp(u^2)du$

...linear problems: (Table 4.1 cont'd)

	$f(t)$	$K(s,f(s))$	$g(t)$	I	ref.
10a.	$t^{\frac{1}{2}}$	$-f^2$	$t^{\frac{1}{2}}+\frac{4}{3}\,t^{3/2}$	[0,2]	
10b.	$t^{\frac{1}{2}}$	$-f^3$	$t^{\frac{1}{2}}+\frac{3}{8}\,\pi t^2$	[0,2]	
10c.	$t^{\frac{1}{2}}$	$-f^4$	$t^{\frac{1}{2}}+\frac{16}{15}\,t^{5/2}$	[0,3]	
11.	$(1-t)t^{\frac{1}{2}}$	$-\exp(s(1-s)^2-f^2)$	$(3-t)t^{\frac{1}{2}}$	[0,4]	[11]
12.	? $(f(t)=O(t^{7/2}),\ t\to 0)$	$-\pi^{-\frac{1}{2}}(f-\sin(s))^3$	0	[0,1]	[9,18]
13.	? $(f(t)=O(t^{\frac{1}{2}}),\ t\to 0)$	$-\pi^{-\frac{1}{2}}(f(s))^4$	$2\pi^{-\frac{1}{2}}t^{\frac{1}{2}}$	[0,1]	[4,7]

TABLE 4.2 $-{}^{10}\log|f(T)-u(T)|$

S C H E M E A

#	values of h			T	m = r = 1			m = r = 2			m = 3, r = 2		
1.	.2	.1	.05	1	4.70	5.53	6.46	5.89	6.74	7.61	8.49	8.60	9.33
2.	.1	.05	.025	3	6.27	6.17	6.57	7.26	8.18	9.09	8.80	9.69	10.59
3a.	.2	.1	.05	1	e x a c t *)			e x a c t			e x a c t		
3b.	.1	.05	.025	2	5.08	5.37	5.76	6.88	7.49	8.10	9.39	9.57	9.87
3c.	.1	.05	.025	2	5.27	5.80	6.37	e x a c t			e x a c t		
3d.	.1	.05	.025	2	6.89	5.29	5.64	6.59	7.38	8.15	10.13	10.72	11.36
3e.	.1	.05	.025	2	4.57	4.97	5.21	6.52	7.46	8.38	8.10	9.07	10.01
5.	.1	.05	.025	2	4.65	5.05	5.49	6.21	6.85	7.49	7.91	8.65	9.30
6.	.2	.1	.05	1	4.08	4.30	4.67	5.18	5.79	6.43	6.63	7.46	8.23
7.	.1	.05	.01	2	4.71	5.14	6.18	6.59	7.24	8.69	8.41	9.01	10.10
8a.	.1	.05	.025	3	5.16	5.67	6.14	7.28	8.20	9.81	8.67	10.41	10.21
8b.	.1	.05	.025	3	5.12	5.67	6.11	6.28	7.84	7.87	7.39	8.16	9.14
10a.	.1	.05	.025	2	4.23	4.84	5.46	7.36	7.92	8.57	7.34	7.99	8.61
10b.	.1	.05	.025	2	4.16	4.75	5.35	7.36	8.77	9.66	7.76	8.55	9.31
10c.	.1	.05	.025	2	4.12	4.69	5.27	*	*	9.51	*	*	*
12.	.1	.05	.025	1	4.40	5.44	6.55	6.37	7.41	8.44	6.42	7.61	8.89
13.	.1	.05	.025	1	4.00	4.66	5.31	6.13	6.95	7.82	6.18	7.00	7.88

S C H E M E B

#	values of h			T	m = r = 1			m = r = 2			m = 3, r = 2		
1.	.2	.1	.05	1	4.65	5.12	5.64	6.62	7.49	8.37	8.77	9.91	11.07
2.	.1	.05	.025	3	5.38	5.79		7.59	8.50		11.01	11.92	
3a.	.2	.1	.05	1	e x a c t			e x a c t			e x a c t		
3b.	.1	.05	.025	2	4.33	4.74	5.18	5.18	5.61	6.05	5.54	6.00	6.43
3c.	.1	.05	.025	2	e x a c t			e x a c t			e x a c t		
3d.	.1	.05	.025	2	5.65	5.27	5.63	7.75	9.13	9.15	7.78	8.50	9.24
3e.	.1	.05	.025	2	4.60	4.96	5.21	e x a c t			e x a c t		
5.	.1	.05	.025	2	4.09	4.51	4.93	4.88	5.31	5.75	5.25	5.69	6.13
6.	.2	.1	.05	1	3.35	3.71	4.12	4.09	4.50	4.92	4.42	4.86	5.30
7.	.1	.05	.01	2									
8a.	.1	.05	.025	3	4.52	4.99	5.45	5.41	5.85	6.29	5.78	6.23	6.67
8b.	.1	.05	.025	3	4.55	4.99	5.44	5.38	5.83	6.28	5.76	6.22	6.67
10a.	.1	.05	.025	2	4.30	4.92	5.53	6.63	7.24	7.82	5.92	6.55	7.16
10b.	.1	.05	.025	2	4.17	4.76	5.36	7.52	8.80	9.06	6.76	7.52	8.27
10c.	.1	.05	.025	2	4.13	4.70	5.28	6.93	7.96	9.02	7.89	8.82	9.72
12.	.1	.05	.025	1	4.41	5.44	6.54	7.84	7.89	8.72	7.55	8.96	9.52
13.	.1	.05	.025	1	4.03	4.68	5.32	6.57	7.59	8.55	6.43	7.36	8.33

*) within 14D machine precision

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